POLITECNICO MILANO 1863

Elettromagnetismo

Elettricità. Corrente. Magnetismo

Maurizio Zani

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Elettromagnetismo

http://www.mauriziozani.it/wp/?p=1128

Elettromagnetismo

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Elettrizzazione: strofinio

il materiale si elettrizza?

carica

- no: **materiale conduttore**

- sì: **materiale isolante**
  - come la plastica (-): **elettrizzazione resinosa**
  - come il vetro (+): **elettrizzazione vetrosa**

tipo diverso: attrazione
stesso tipo: repulsione

Elettrizzazione: strofinio

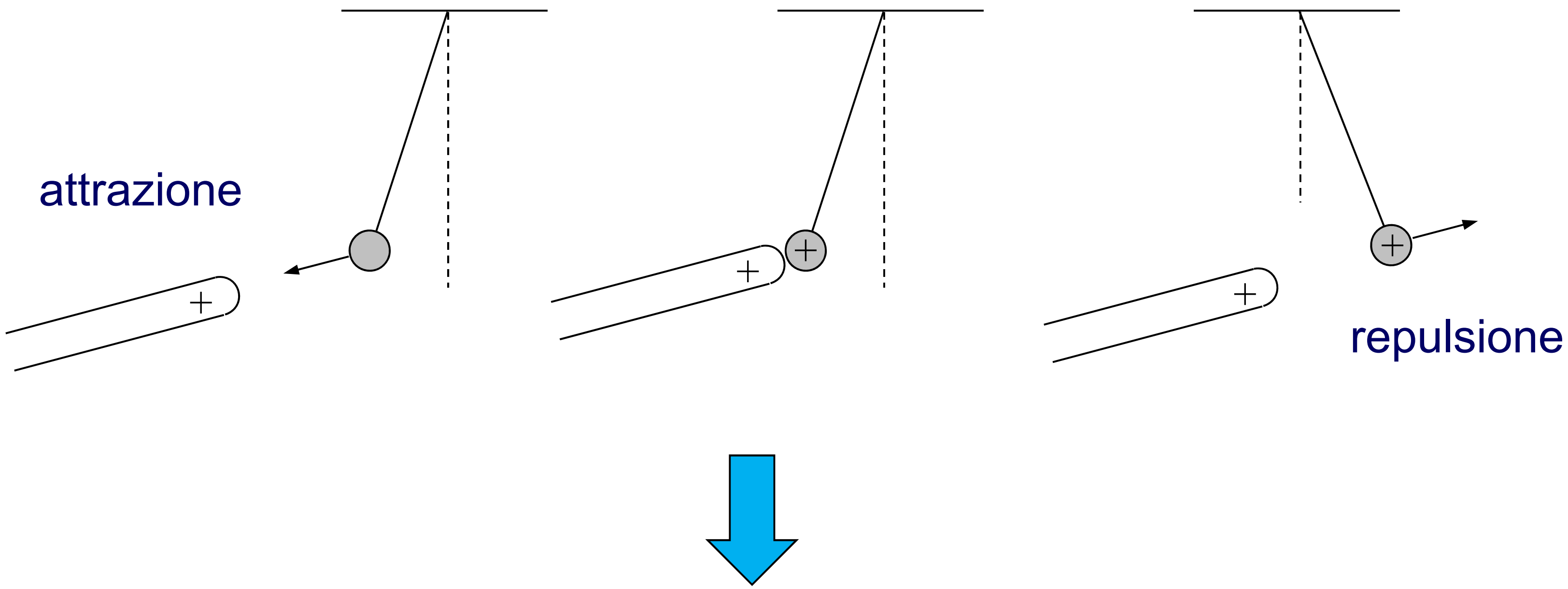
vetro (+)

plastica (-)

Si	H	C
elettronegatività		
1.90	2.20	2.55

Elettrizzazione: contatto

materiali conduttori

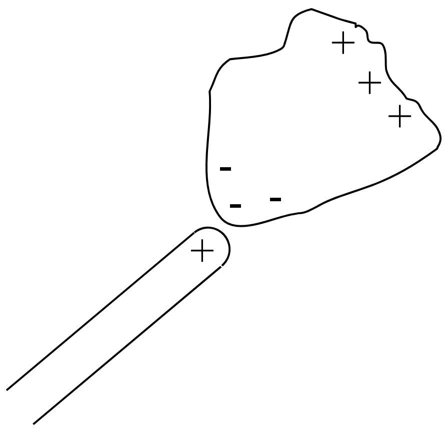


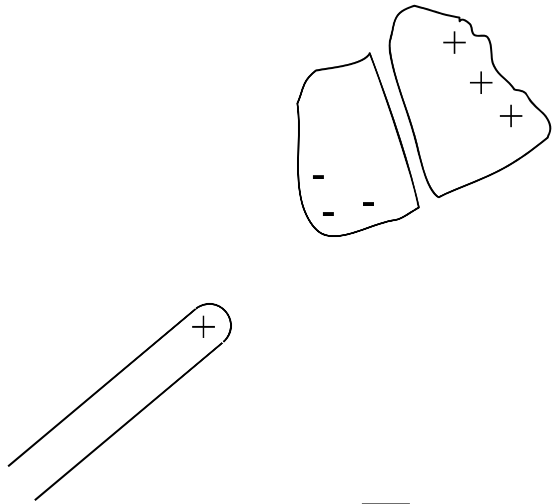
contatto

l'interazione
cambia segno

Elettrizzazione: induzione elettrostatica

materiali conduttori





prossimità (senza contatto)

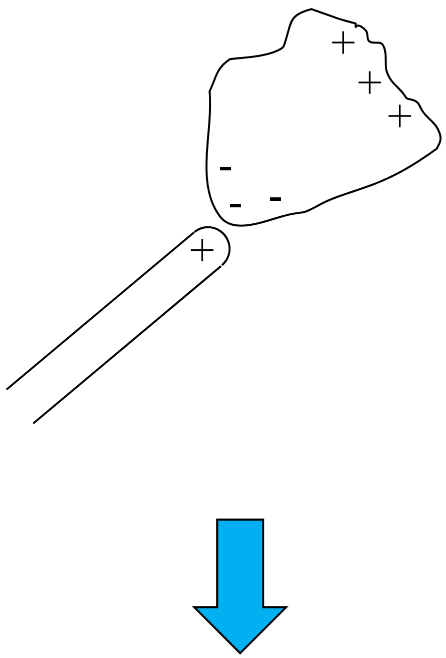
elettrizzazione
localizzata e temporanea

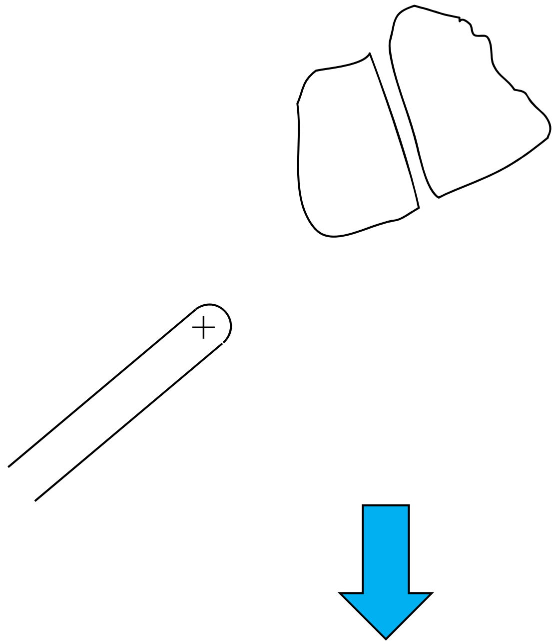
prossimità e taglio

elettrizzazione
opposta e permanente

Elettrizzazione: polarizzazione

materiali isolanti





prossimità (senza contatto)

elettrizzazione
localizzata e temporanea

prossimità e taglio

nessuna
elettrizzazione

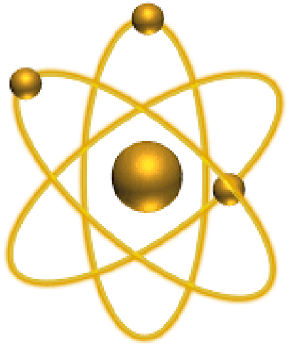
Forza elettrica: carica elettrica

q

carica elettrica

$$[q] = [I][t] = \mathrm{As} = \mathrm{C}$$

coulomb



protone

- $m_p = 1.672622 \cdot 10^{-27}$ kg; $q_p = 1.602176 \cdot 10^{-19}$ C

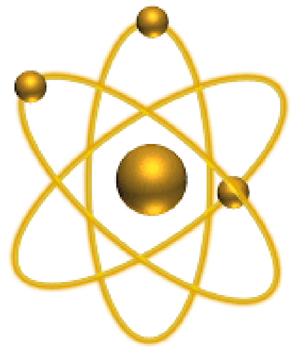
elettrone

- $m_e = 9.109382 \cdot 10^{-31}$ kg; $q_e = -1.602176 \cdot 10^{-19}$ C

neutrone

- $m_n = 1.674927 \cdot 10^{-27}$ kg; $q_n = 0$ C

Forza elettrica: struttura della materia



Modello di Thomson (1902)
- atomo come sfera carica positivamente (e senza massa)
- elettroni cariche negative al suo interno (con massa)

Modello di Lorentz (1905)
- nucleo carico positivamente
- elettroni come sfera carica negativamente

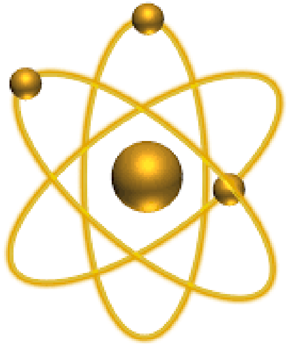
Modello di Rutherford (1911)
- nucleo carico positivamente
- elettroni cariche negative che orbitano

Modello di Bohr (1913)
- nucleo carico positivamente
- elettroni cariche negative su orbite stazionarie

Forza elettrica: struttura della materia

Atomo

- atomo: $r = 10^{-10}$ m
- nucleo: $r = 10^{-15}$ m
- elettrone: $r = 10^{-18}$ m



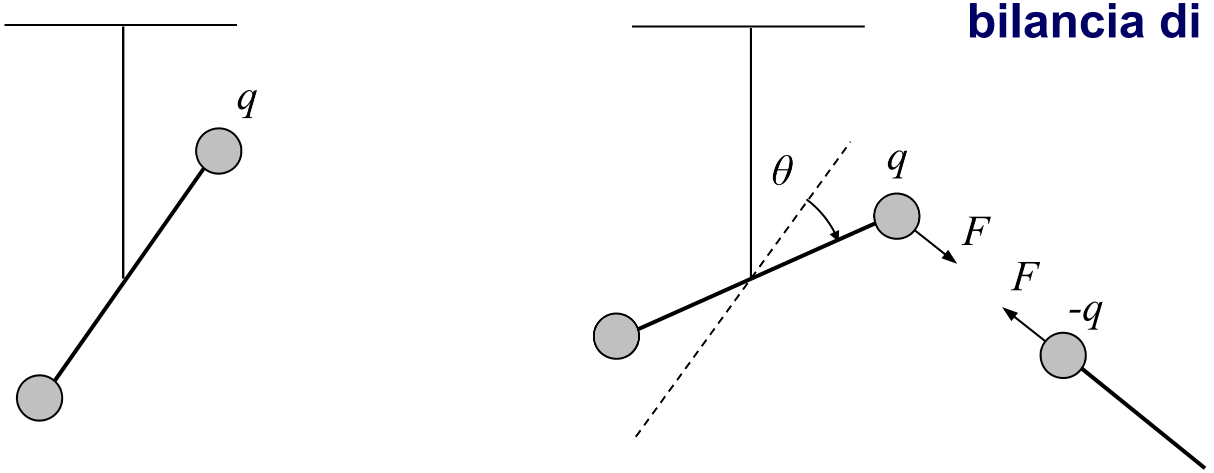
$\cdot 10^{17}$

- atomo => 10^{7} m (pianeta Terra)
- nucleo => 10^{2} m (campo da calcio)
- elettrone => 10^{-1} m (pallone da calcio)

l'atomo è vuoto!

Forza elettrica: forza di Coulomb

bilancia di torsione



$$\begin{cases} M_t = k_t\theta \\ M_e = F_e b \end{cases}$$

$$F_e = \frac{k_t\theta}{b}$$

costante elettrica

$$\vec{F}_e = k_e \frac{q_1 q_2}{r^2}\vec{u}_r$$

$$k_e = \frac{1}{4\pi\varepsilon_0} = 8.9874\cdot 10^9 \ \mathrm{Nm^2/C^2}$$

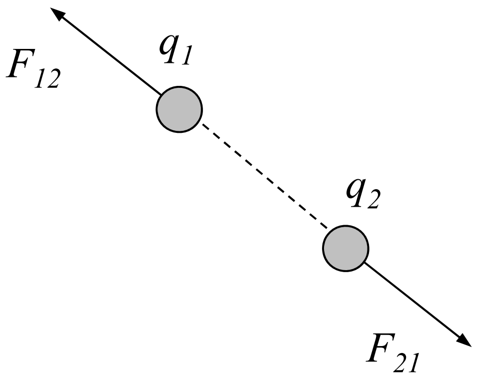
forza elettrica (di Coulomb)

$$\varepsilon_0 = 8.85418781762\cdot 10^{-12} \ \mathrm{C^2/Nm^2}$$

forza fondamentale

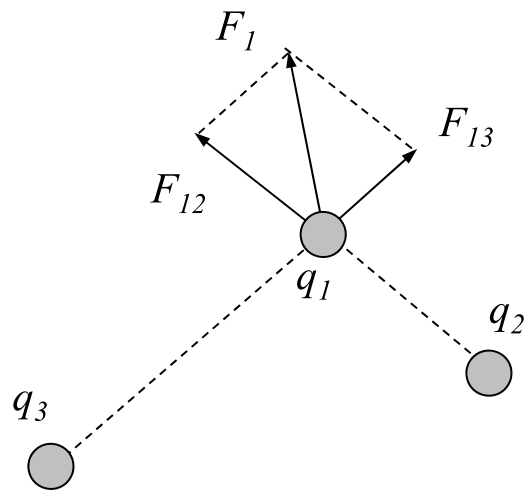
permittività elettrica

Forza elettrica: forza di Coulomb



azione e reazione

$$\left|\vec{F}_{12}\right| = \left|\vec{F}_{21}\right|$$



sovrapposizione degli effetti

$$\vec{F}_1 = \vec{F}_{12} + \vec{F}_{13}$$

Forza elettrica: forza di Coulomb

$$\vec{F}_g = -\gamma \frac{m_p m_e}{r^2} \vec{u}_r$$

$$\frac{F_e}{F_g} = \frac{k_e}{\gamma} \frac{q_p q_e}{m_p m_e} \approx 10^{39}$$

- età dell'universo
  - 13 M anni = $4*10^{17}$ s
- dimensione dell'universo
  - 92 M anni luce = $8.7*10^{26}$ m

confronto con la
forza gravitazionale

$$\vec{F}_e = k_e \frac{q_1 q_2}{r^2} \vec{u}_r$$

forza elettrica (di Coulomb)

interazione tra due cariche
q = 1 C; r = 1 m

$$F_e = 9 \cdot 10^9 \text{ N}$$

450 Shuttle!

Campo elettrico

$$\vec{F}_e = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\vec{u}_r$$

$$[E] = \frac{[F]}{[q]} = \frac{\mathrm{N}}{\mathrm{C}}$$

$$\vec{F}_e = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\vec{u}_r = q_1\left(\frac{1}{4\pi\varepsilon_0}\frac{q_2}{r^2}\vec{u}_r\right) = q_1\vec{E}_2$$

oggetto

effetto

causa

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\vec{u}_r$$

campo elettrico

$$\vec{F}_1 = \sum \vec{F}_i = \sum q_1\vec{E}_i = q_1\sum \vec{E}_i = q_1\vec{E}$$

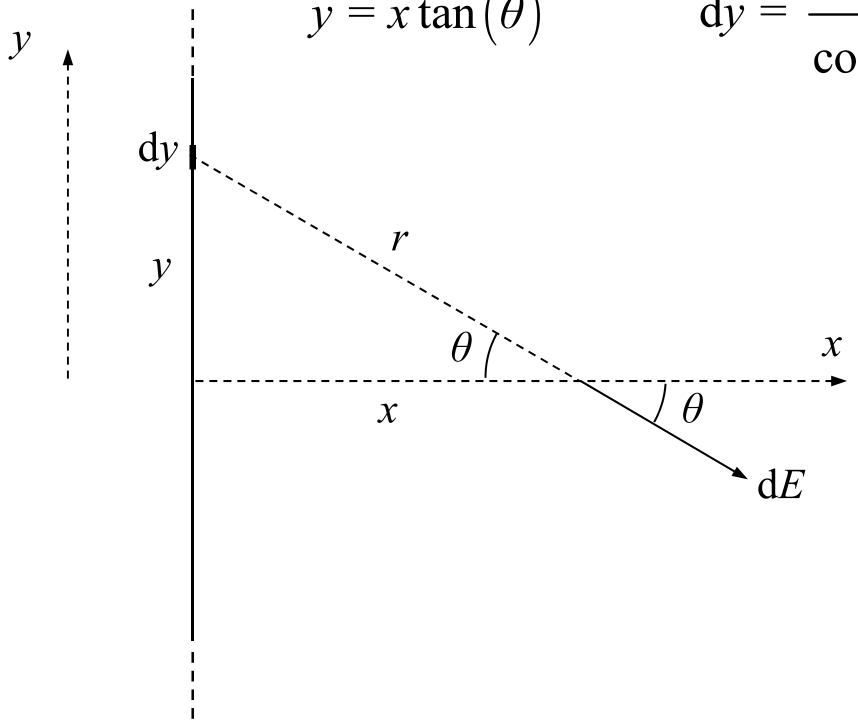
vale la sovrapposizione degli effetti

$$\vec{E} = \sum \vec{E}_i$$

$$\vec{E} = \int \mathrm{d}\vec{E}$$

Campo elettrico

filo rettilineo infinito unif. carico



$$y = x\tan(\theta) \qquad dy = \frac{x}{\cos^2(\theta)}d\theta$$

$$dq = \lambda\, dy = \lambda \frac{x}{\cos^2(\theta)}d\theta$$

$$dE_x = dE\cos(\theta) = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}\cos(\theta) =$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{\lambda}{x}\cos(\theta)\, d\theta$$

$$r = \frac{x}{\cos(\theta)}$$

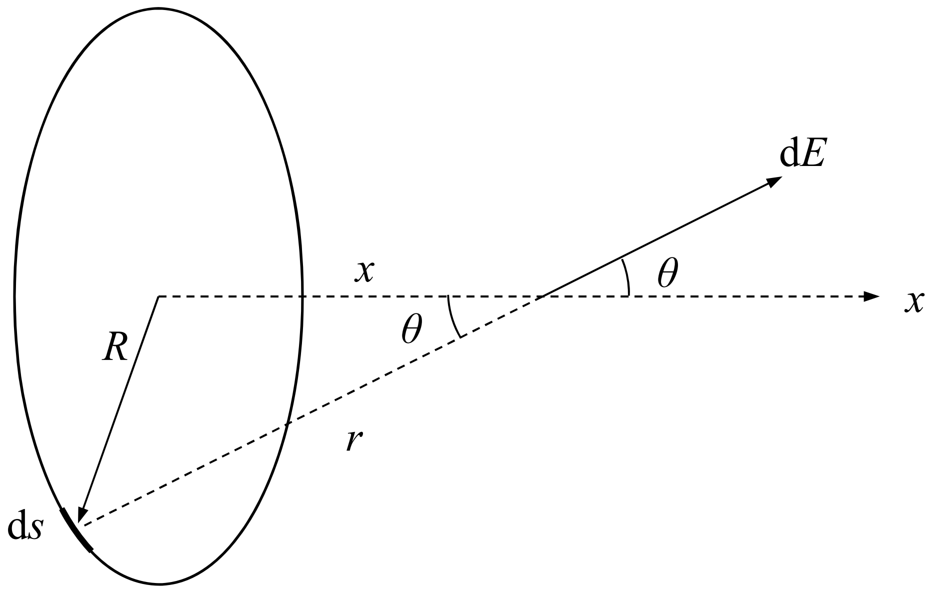
$$E = \int dE_x = \frac{1}{4\pi\varepsilon_0}\frac{\lambda}{x}\int_{-\pi/2}^{+\pi/2}\cos(\theta)\, d\theta = \frac{1}{4\pi\varepsilon_0}\frac{\lambda}{x}\left[\sin(\theta)\right]_{-\pi/2}^{+\pi/2} = \frac{1}{2\pi\varepsilon_0}\frac{\lambda}{x}$$

Campo elettrico

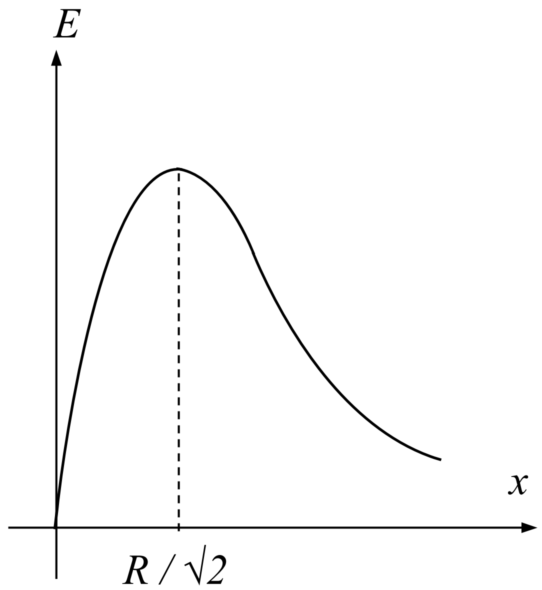
anello unif. carico

$$\mathrm{d}E_x = \mathrm{d}E\cos(\theta) = \frac{1}{4\pi\varepsilon_0}\frac{\mathrm{d}q}{x^2+R^2}\cos(\theta) =$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{\mathrm{d}q}{x^2+R^2}\frac{x}{\sqrt{x^2+R^2}}$$



$$E = \int \mathrm{d}E_x = \frac{1}{4\pi\varepsilon_0}\frac{x\,q}{\left(x^2+R^2\right)^{3/2}}$$

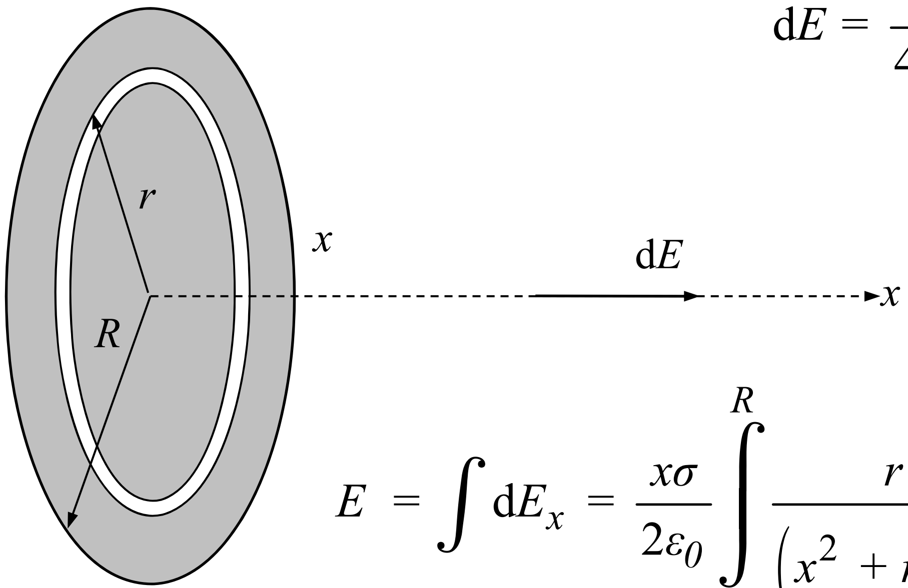


Campo elettrico

disco unif. carico

$$\sigma = \frac{q_{disco}}{\pi R^2} \qquad q_{disco} = \sigma \pi r^2 \qquad dq = \sigma\, 2\pi r\, dr$$

$$dE = \frac{1}{4\pi\varepsilon_0} \frac{x\, dq}{\left(x^2 + r^2\right)^{3/2}}$$



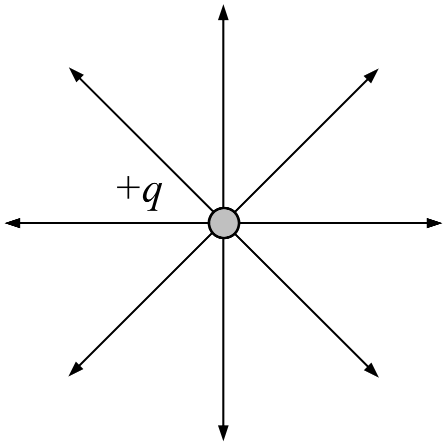
$$E = \int dE_x = \frac{x\sigma}{2\varepsilon_0} \int_0^R \frac{r}{\left(x^2 + r^2\right)^{3/2}} dr = \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{x}{\sqrt{x^2 + R^2}}\right)$$

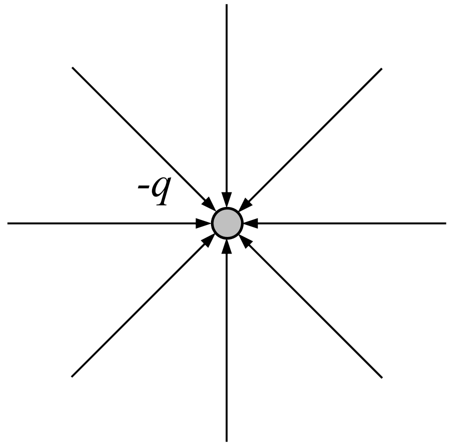
$$x \ll R: \quad E \approx \frac{\sigma}{2\varepsilon_0}$$

$$x \gg R: \quad E \approx \frac{1}{4\pi\varepsilon_0} \frac{q}{x^2}$$

Campo elettrico: linee di flusso

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\vec{u}_r \qquad \vec{E} = \int d\vec{E}$$



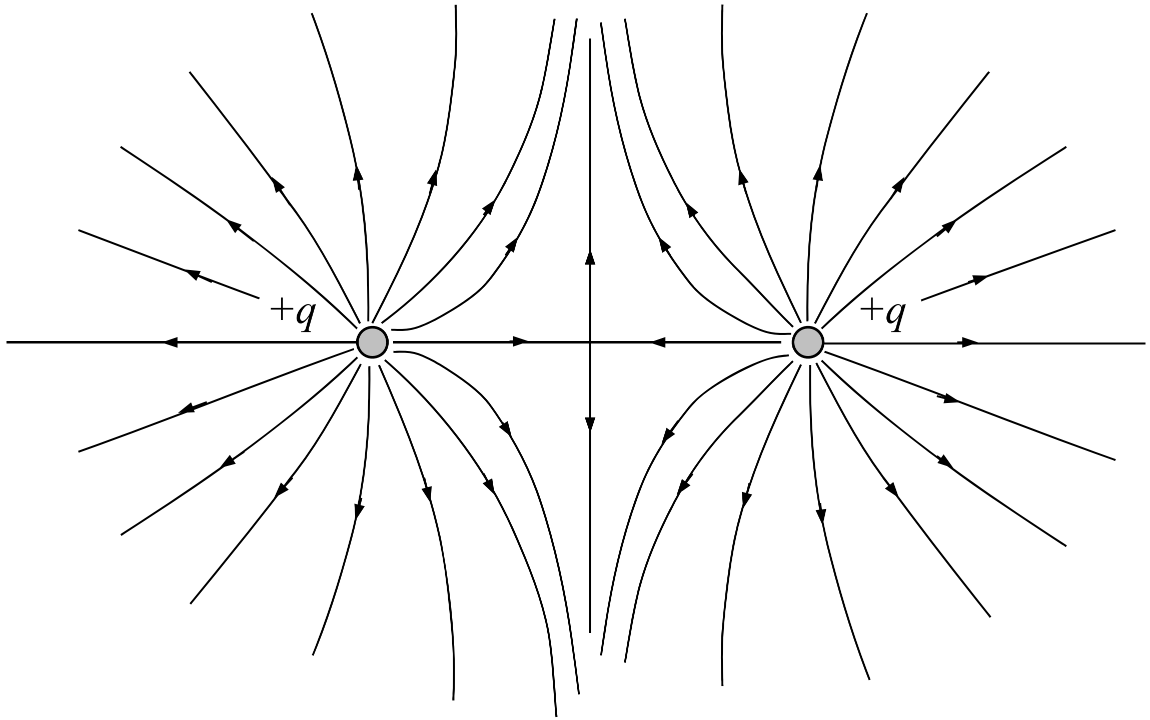


Linee di flusso

- linee orientate, tangenti (direzione) e concordi (verso) al campo
- si addensano dove il campo è più intenso
- non si incrociano mai
- partono (sorgente) e terminano (pozzo) sulle cariche o all'infinito

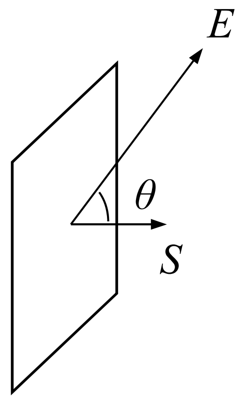
Campo elettrico: linee di flusso

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \vec{u}_r \qquad \vec{E} = \int \mathrm{d}\vec{E}$$



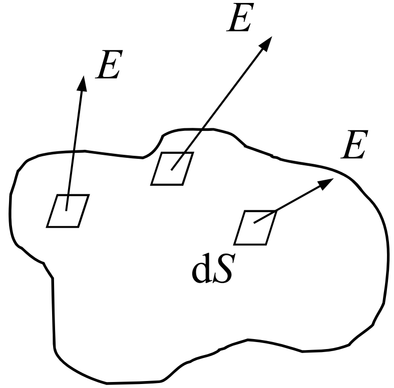
Teorema di Gauss: flusso

campo omogeneo: $\Phi\left(\vec{E}\right) = \vec{E}\cdot\vec{S}$ $\quad \left[\Phi\left(\vec{E}\right)\right] = [E][S] = \dfrac{\text{N}}{\text{C}}\text{m}^2$



flusso

campo/superficie variabile: $\Phi\left(\vec{E}\right) = \int \mathrm{d}\Phi\left(\vec{E}\right) = \int \vec{E}\cdot \mathrm{d}\vec{S}$

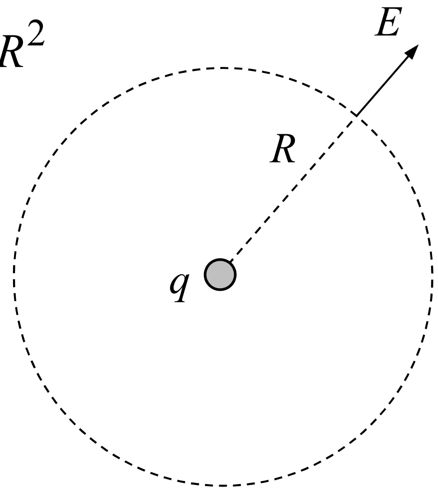


- flusso additivo tra campi $\Phi\left(\vec{E}\right) = \int \left(\vec{E}_1 + \vec{E}_2\right)\cdot \mathrm{d}\vec{S} = \int \vec{E}_1\cdot \mathrm{d}\vec{S} + \int \vec{E}_2\cdot \mathrm{d}\vec{S} = \Phi_1\left(\vec{E}\right) + \Phi_2\left(\vec{E}\right)$

- flusso additivo in superfici $\Phi\left(\vec{E}\right) = \int_{S_3} \vec{E}\cdot \mathrm{d}\vec{S} = \int_{S_1} \vec{E}\cdot \mathrm{d}\vec{S} + \int_{S_2} \vec{E}\cdot \mathrm{d}\vec{S} = \Phi_1\left(\vec{E}\right) + \Phi_2\left(\vec{E}\right)$

Teorema di Gauss: superficie sferica

$$S = 4\pi R^2$$



$$\Phi\left(\vec{E}\right) = \oint \vec{E} \cdot \mathrm{d}\vec{S} = \oint E \,\mathrm{d}S = E \oint \mathrm{d}S = E\, S$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \vec{u}_r$$

$$\Phi\left(\vec{E}\right) = \oint \vec{E} \cdot \mathrm{d}\vec{S} = \frac{q}{\varepsilon_0}$$

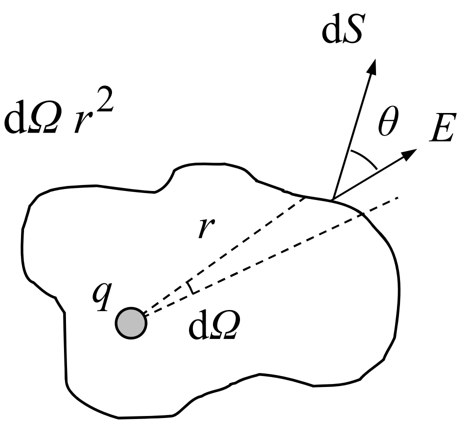
teorema di Gauss

$$\Phi\left(\vec{E}\right) = \oint \vec{E} \cdot \mathrm{d}\vec{S} = 4\pi k_e \cdot q$$

$$\Phi\left(\vec{G}\right) = \oint \vec{G} \cdot \mathrm{d}\vec{S} = -4\pi\gamma \cdot m$$

Teorema di Gauss: superficie generica

$$\mathrm{d}S\cos(\theta) = \mathrm{d}S' = \mathrm{d}\Omega\, r^2$$



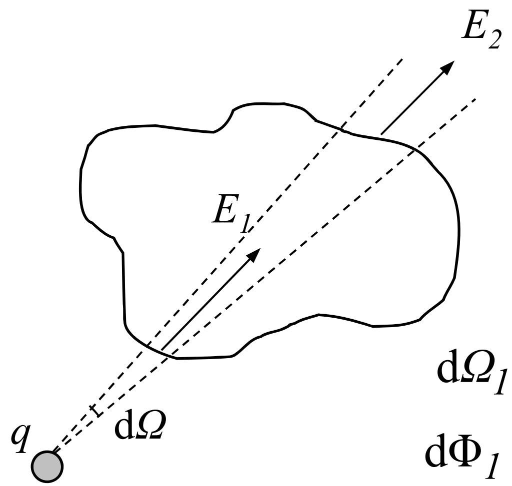
$$\Phi(\vec{E}) = \oint \vec{E}\cdot \mathrm{d}\vec{S} = \oint E\, \mathrm{d}S\, \cos(\theta)$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\vec{u}_r$$

$$\Phi(\vec{E}) = \oint \mathrm{d}\Phi = \frac{q}{4\pi\varepsilon_0}\oint \mathrm{d}\Omega$$

carica interna $\oint \mathrm{d}\Omega = 4\pi$

carica esterna $\oint \mathrm{d}\Phi = 0$



$$\mathrm{d}\Omega_1 = \mathrm{d}\Omega_2$$

$$\mathrm{d}\Phi_1 = -\mathrm{d}\Phi_2$$

Teorema di Gauss

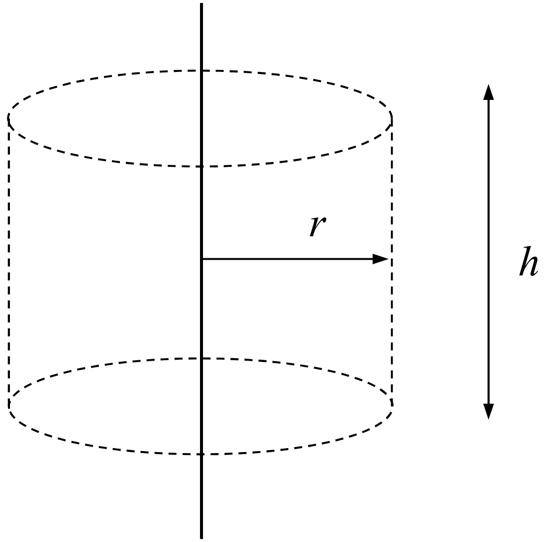
"Il flusso del campo elettrico attraverso una superficie chiusa dipende unicamente dalla carica netta contenuta nella superficie, e ne risulta proporzionale secondo un fattore $1/\varepsilon_0$"

$$\Phi\left(\vec{E}\right) = \oint \vec{E} \cdot \mathrm{d}\vec{S} = \frac{q_{int}}{\varepsilon_0}$$

sempre valido, non sempre utile

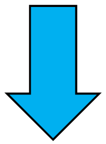
Teorema di Gauss

filo rettilineo infinito unif. carico



per simmetria, il campo elettrico è

- radiale rispetto al filo
- invariante per traslazione lungo il filo
- invariante per rotazione attorno al filo



simmetrica cilindrica

$$\Phi\left(\vec{E}\right) = \underbrace{\oint \vec{E}\cdot \mathrm{d}\vec{S}}_{\text{flusso}} = \underbrace{\frac{q_{int}}{\varepsilon_0}}_{\text{Gauss}}$$

$$\Phi\left(\vec{E}\right) = \oint \vec{E}\cdot \mathrm{d}\vec{S} = \int_{\text{lato}} \vec{E}\cdot \mathrm{d}\vec{S} + \cancel{\int_{\text{basi}} \vec{E}\cdot \mathrm{d}\vec{S}} =$$

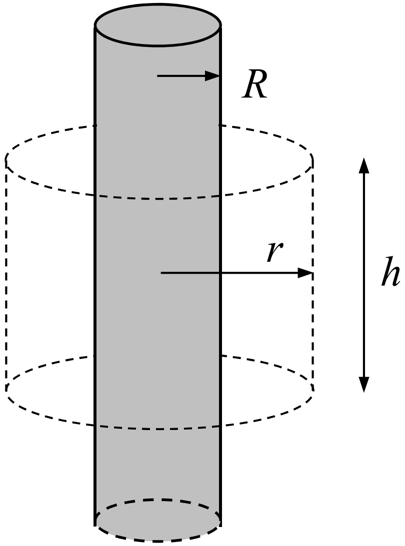
$$= \int_{\text{lato}} E\,\mathrm{d}S = E \int_{\text{lato}} \mathrm{d}S = E\,2\pi r\,h =$$

$$= \frac{q_{int}}{\varepsilon_0} = \frac{\lambda h}{\varepsilon_0}$$

$$E = \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{r}$$

Teorema di Gauss

cilindro rettilineo infinito unif. carico



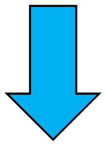
$$\Phi\left(\vec{E}\right) = \oint \vec{E}\cdot d\vec{S} = E\,2\pi r\,h = \frac{q_{int}}{\varepsilon_0}$$

$r > R$: $\quad q_{int} = \rho V = \rho\pi R^2 h \qquad E = \dfrac{\rho R^2}{2\varepsilon_0}\dfrac{1}{r}$

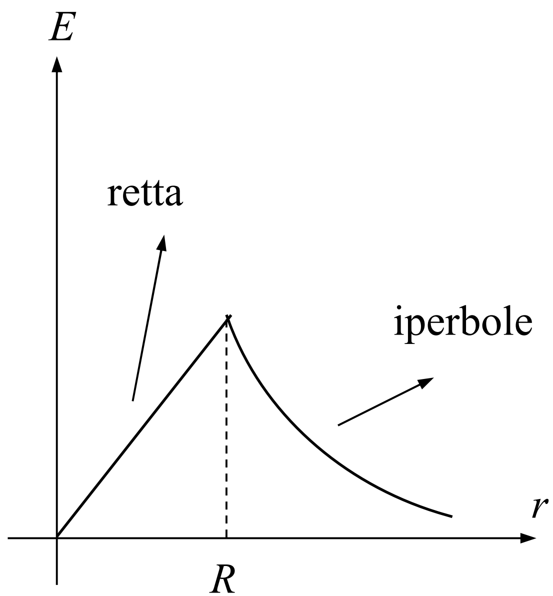
$r < R$: $\quad q_{int} = \rho V = \rho\pi r^2 h \qquad E = \dfrac{\rho}{2\varepsilon_0} r$

per simmetria, il campo elettrico è

- radiale rispetto al filo
- invariante per traslazione lungo il filo
- invariante per rotazione attorno al filo



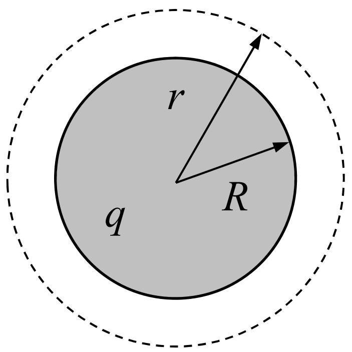
simmetrica cilindrica



Teorema di Gauss

sfera unif. carica

$$\Phi\left(\vec{E}\right) = \oint \vec{E} \cdot \mathrm{d}\vec{S} = E\, 4\pi r^2 = \frac{q_{int}}{\varepsilon_0}$$



$r > R$: $\quad q_{int} = q \qquad E = \dfrac{1}{4\pi\varepsilon_0} \dfrac{q}{r^2}$

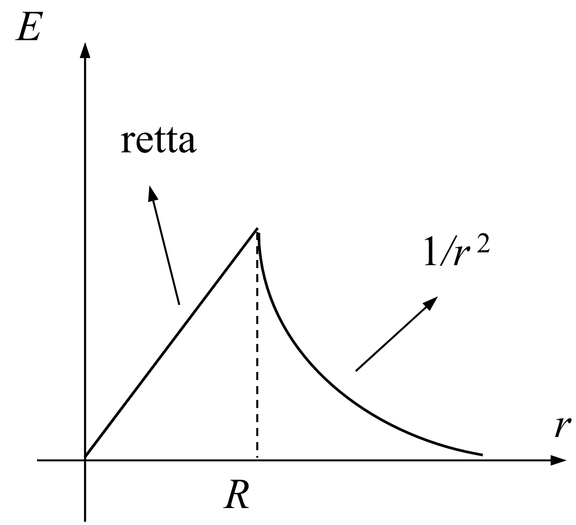
$r < R$: $\quad q_{int} = \rho V = \dfrac{q}{\frac{4}{3}\pi R^3} \frac{4}{3}\pi r^3 = \dfrac{q}{R^3} r^3 \qquad E = \dfrac{1}{4\pi\varepsilon_0} \dfrac{q}{R^3} r$

per simmetria, il campo elettrico è

- radiale rispetto al centro
- invariante per rotazione della sfera

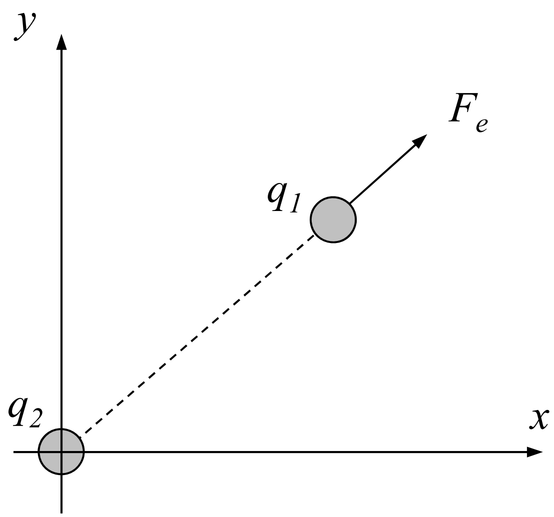
⇓

simmetrica sferica



Campo conservativo: energia potenziale

$$\vec{F}_e = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\vec{u}_r \qquad \mathrm{d}\vec{r} = \mathrm{d}r\,\vec{u}_r + r\,\mathrm{d}\theta\,\vec{u}_\theta$$



$$W = \int_A^B \vec{F}_e \cdot \mathrm{d}\vec{r} = \int_{r_A}^{r_B} \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\,\mathrm{d}r$$

$$U_e = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r}$$

energia potenziale della forza elettrica

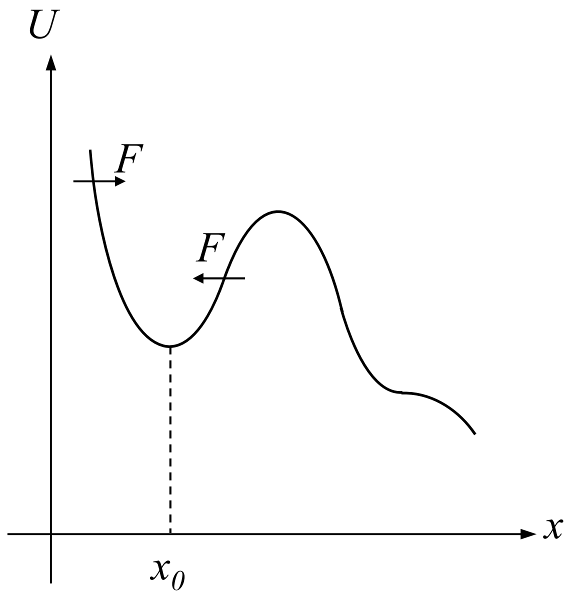
$$W = \int_{r_A}^{r_B} \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\,\mathrm{d}r = -\int_A^B \mathrm{d}\left(\frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r}\right) = -\int_A^B \mathrm{d}U_e = -\Delta U_e$$

Campo conservativo: energia potenziale

$$U(\vec{r}) = U_0 - \int_{\vec{r}_0}^{\vec{r}} \vec{F} \cdot d\vec{r}$$

$\Rightarrow$

$$\vec{F} = ?$$



$$\begin{cases} đW = \vec{F} \cdot d\vec{r} = F_x dx + F_y dy + F_z dz = -dU \\ dU = \dfrac{\partial U}{\partial x} dx + \dfrac{\partial U}{\partial y} dy + \dfrac{\partial U}{\partial z} dz \end{cases}$$

$\Rightarrow$

$$\begin{cases} F_x = -\dfrac{\partial U}{\partial x} \\ F_y = -\dfrac{\partial U}{\partial y} \\ F_z = -\dfrac{\partial U}{\partial z} \end{cases}$$

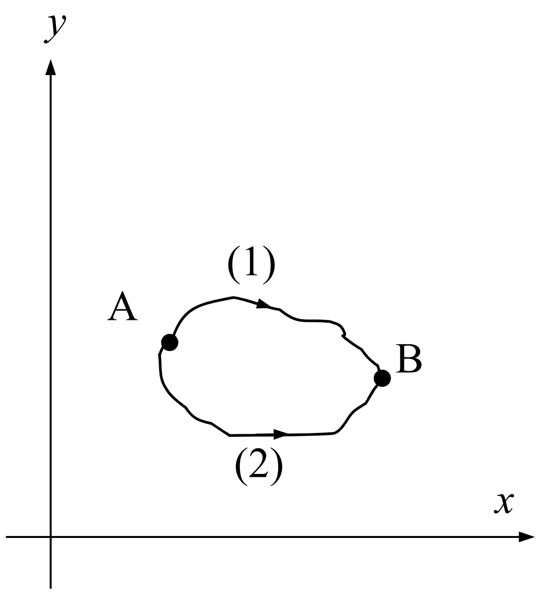
$$\vec{F} = -\,\mathrm{grad}(U)$$

gradiente

Campo conservativo: energia potenziale

$$W = \int_{(1)\,A}^{B} \vec{F}\cdot \mathrm{d}\vec{r} = \int_{(2)\,A}^{B} \vec{F}\cdot \mathrm{d}\vec{r}$$

$$W = \oint \vec{F}\cdot \mathrm{d}\vec{r} = \int_{(1)\,A}^{B} \vec{F}\cdot \mathrm{d}\vec{r} + \int_{(2)\,B}^{A} \vec{F}\cdot \mathrm{d}\vec{r} = \int_{(1)\,A}^{B} \vec{F}\cdot \mathrm{d}\vec{r} - \int_{(2)\,A}^{B} \vec{F}\cdot \mathrm{d}\vec{r} = 0$$



$$\vec{F} = q\vec{E}$$

$$\Lambda\left(\vec{F}\right) = \oint \vec{F}\cdot \mathrm{d}\vec{r} = q\oint \vec{E}\cdot \mathrm{d}\vec{r} = 0$$

circuitazione

III legge di Maxwell

"La circuitazione del campo elettrico lungo una linea chiusa
è nulla"

$$\Lambda\left(\vec{E}\right) = \oint \vec{E} \cdot \mathrm{d}\vec{r} = 0$$

Campo conservativo: potenziale elettrico

$$U_e = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r}$$

$$[V] = \frac{[U]}{[q]} = \frac{\mathrm{J}}{\mathrm{C}} = \mathrm{V}$$

volt

oggetto

$$U_e = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r} = q_1\left(\frac{1}{4\pi\varepsilon_0}\frac{q_2}{r}\right) = q_1 V_2$$

effetto

causa

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

potenziale elettrico

$$U_1 = \sum U_i = \sum q_1 V_i = q_1 \sum V_i = q_1 V$$

vale la sovrapposizione degli effetti

$$\begin{cases} V = \sum V_i \\ V = \int \mathrm{d}V \end{cases}$$

Campo conservativo: potenziale elettrico

$$\vec{F} = -\,\text{grad}\left(U\right)$$

$$\begin{cases} F_x = -\dfrac{\partial U}{\partial x} \\ F_y = -\dfrac{\partial U}{\partial y} \\ F_z = -\dfrac{\partial U}{\partial z} \end{cases}$$

$$\vec{F} = q\vec{E} \qquad\qquad U = qV$$

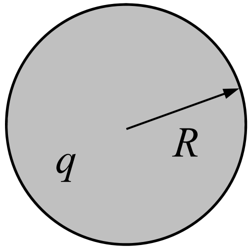
$$\vec{E} = -\,\text{grad}\left(V\right)$$

$$\begin{cases} E_x = -\dfrac{\partial V}{\partial x} \\ E_y = -\dfrac{\partial V}{\partial y} \\ E_z = -\dfrac{\partial V}{\partial z} \end{cases}$$

$$\left[E\right] = \frac{\left[F\right]}{\left[q\right]} = \frac{\text{N}}{\text{C}} = \frac{\text{V}}{\text{m}}$$

Campo conservativo: potenziale elettrico

sfera unif. carica



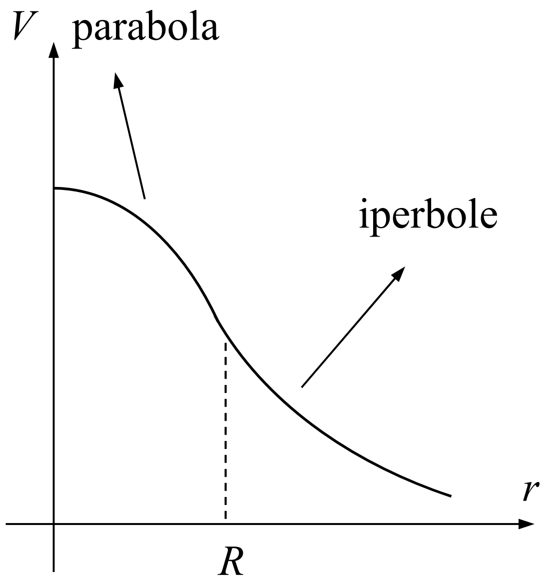
$$r > R: \quad E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}$$

$$r < R: \quad E = \frac{1}{4\pi\varepsilon_0}\frac{q}{R^3}\,\mathrm{r}$$

$$E_r = -\frac{\partial V}{\partial r}$$

$$r > R: \quad V(r) = \cancel{V_\infty} - \int_{r_\infty}^{r} E\,\mathrm{d}r = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

$$r < R: \quad V(r) = V_R - \int_{R}^{r} E\,\mathrm{d}r = \frac{1}{8\pi\varepsilon_0}\frac{q}{R}\left(3 - \frac{r^2}{R^2}\right)$$



Campo conservativo: energia elettrica

$$U_e = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r}$$

$$W_{int} = \int_A^B \vec{F} \cdot d\vec{r} = -\Delta U$$

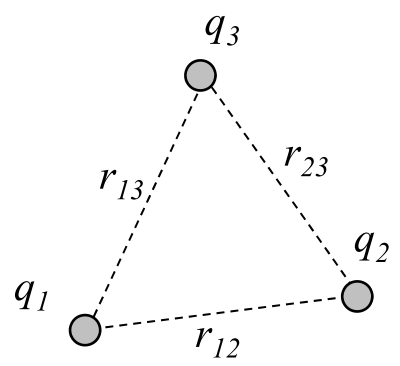
$$W_{ext} = \int_A^B \vec{F}_{ext} \cdot d\vec{r} = \int_A^B -\vec{F} \cdot d\vec{r} = -W_{int} = \Delta U = U_f - U_i$$

$\vec{r}_i = \infty$ $\quad U_i = 0 \quad \Rightarrow \quad W_{ext} = U_f$

Campo conservativo: energia elettrica

$$U_e = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r}$$

$$W_1 = 0$$



$$W_2 = \Delta U_2 = q_2 \Delta V = q_2 V_1 = q_2 \frac{1}{4\pi\varepsilon_0}\frac{q_1}{r_{12}}$$

$$W_3 = \Delta U_3 = q_3 \Delta V = q_3 \left(V_1 + V_2\right) = q_3 \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{r_{13}} + \frac{q_2}{r_{23}}\right)$$

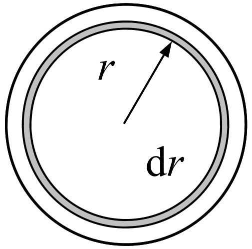
$$E_e = W = W_1 + W_2 + W_3 = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right)$$

energia elettrica

$$E_e = \frac{1}{2}\sum_{i \neq j}^{n} U_{ij} \qquad E_e = \frac{1}{2}\sum_{i=1}^{n} q_i V_i \qquad \Rightarrow \qquad E_e = \frac{1}{2}\int V\,\mathrm{d}q$$

Campo conservativo: energia elettrica

sfera unif. carica



$$r > R: \quad V = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

$$r < R: \quad V = \frac{1}{8\pi\varepsilon_0}\frac{q}{R}\left(3 - \frac{r^2}{R^2}\right)$$

$$E_e = \frac{1}{2}\int V\,\mathrm{d}q = \frac{1}{2}\int_0^R \frac{1}{8\pi\varepsilon_0}\frac{q}{R}\left(3 - \frac{r^2}{R^2}\right)\rho\, 4\pi r^2 \mathrm{d}r = \frac{3q^2}{20\pi\varepsilon_0 R}$$

$$\mathrm{d}q = \rho\,\mathrm{d}V = \rho\, 4\pi r^2 \mathrm{d}r \qquad \rho = \frac{q}{\frac{4}{3}\pi R^3}$$

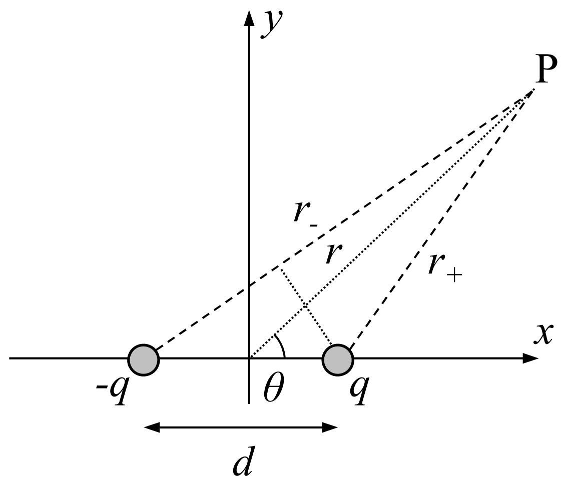
e se tutta la carica andasse sulla superficie?

$$E_e = \frac{1}{2}Vq = $$

$$= \frac{1}{2}\frac{q}{4\pi\varepsilon_0 R}q = \frac{q^2}{8\pi\varepsilon_0 R}$$

Dipolo elettrico: interazioni create

$$V_+ = \frac{1}{4\pi\varepsilon_0}\frac{q}{r_+} \qquad V_- = \frac{1}{4\pi\varepsilon_0}\frac{-q}{r_-}$$



$$V = V_+ + V_- = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_+} + \frac{-1}{r_-}\right) = \frac{q}{4\pi\varepsilon_0}\frac{r_- - r_+}{r_- r_+}$$

approssimazione di dipolo

$$r \gg d \qquad \begin{cases} r_- - r_+ \approx d\cos(\theta) \\ r_- r_+ \approx r^2 \end{cases}$$

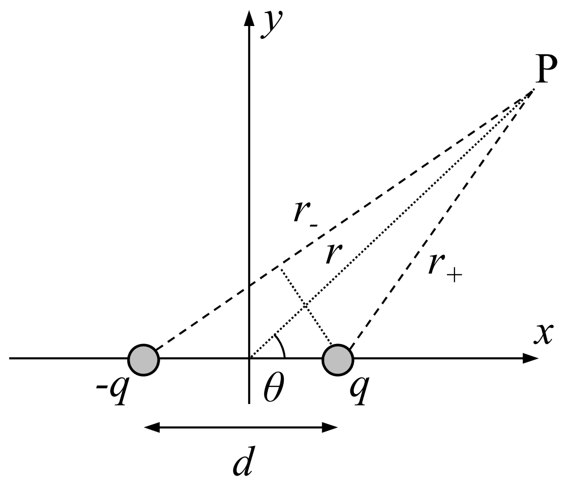
$$\vec{p} = q\vec{d} \qquad [p] = [q][d] = \mathrm{Cm}$$

momento di dipolo elettrico

$$V = \frac{q}{4\pi\varepsilon_0}\frac{r_- - r_+}{r_- r_+} \approx \frac{1}{4\pi\varepsilon_0}\frac{qd\cos(\theta)}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{p\cos(\theta)}{r^2}$$

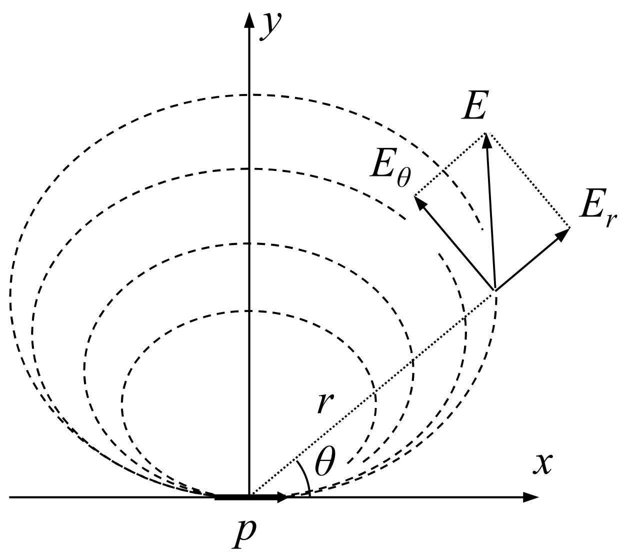
Dipolo elettrico: interazioni create

$$V = \frac{1}{4\pi\varepsilon_0}\frac{p\cos(\theta)}{r^2}$$



$$\vec{E} = -\,\mathrm{grad}(V) \qquad \begin{cases} E_r = -\dfrac{\mathrm{d}V}{\mathrm{d}r} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{2p\cos(\theta)}{r^3} \\ E_\theta = -\dfrac{1}{r}\dfrac{\mathrm{d}V}{\mathrm{d}\theta} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{p\sin(\theta)}{r^3} \end{cases}$$

$$\begin{cases} \theta = 0: \quad E_\theta = 0 \qquad E_r = \dfrac{1}{4\pi\varepsilon_0}\dfrac{2p}{r^3} \\ \theta = \dfrac{\pi}{2}: \quad E_\theta = \dfrac{1}{4\pi\varepsilon_0}\dfrac{p}{r^3} \qquad E_r = 0 \end{cases}$$

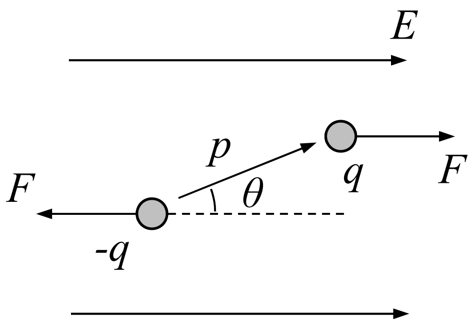


Dipolo elettrico: interazioni subite

$$\vec{M} = \vec{d} \times \vec{F} = \vec{d} \times q\vec{E} = q\vec{d} \times \vec{E} = \vec{p} \times \vec{E}$$

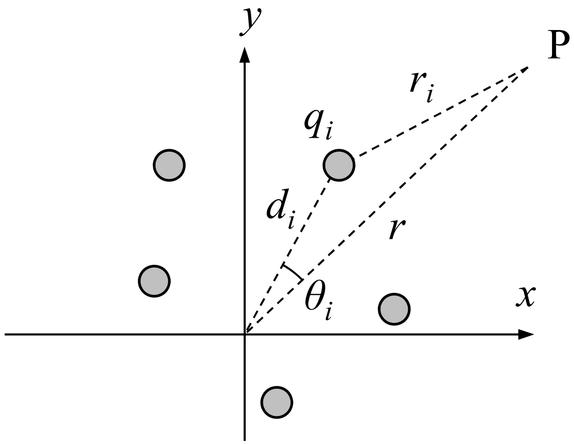
$$U = qV_+ - qV_- = q\left(V_+ - V_-\right) = q\left(-E\,\Delta x\right) =$$

$$= q\left(-E\,d\cos\left(\theta\right)\right) = -pE\cos\left(\theta\right) = -\vec{p}\cdot\vec{E}$$



$$\vec{F} = -\,\mathrm{grad}\left(U\right) = \mathrm{grad}\left(\vec{p}\cdot\vec{E}\right)$$

Dipolo elettrico: sviluppo in multipoli



$$V_i = \frac{1}{4\pi\varepsilon_0}\frac{q_i}{r_i}$$

$$r_i = \left|\vec{r} - \vec{d}_i\right| = \sqrt{r^2 + d_i^2 - 2rd_i\cos(\theta_i)} =$$

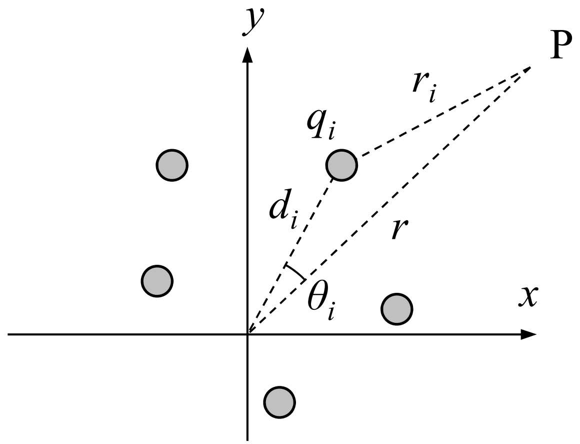
$$= r\sqrt{1 + \left(\frac{d_i}{r}\right)^2 - 2\frac{d_i}{r}\cos(\theta_i)}$$

$$V_i = \frac{1}{4\pi\varepsilon_0}\frac{q_i}{r\sqrt{1 + \boxed{\left(\frac{d_i}{r}\right)^2 - 2\frac{d_i}{r}\cos(\theta_i)}}} \approx \qquad \to x$$

$$r \gg d_i: \quad \frac{1}{\sqrt{1+x}} \approx 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \ldots$$

$$\approx \frac{1}{4\pi\varepsilon_0}\frac{q_i}{r}\left[1 + \cos(\theta_i)\left(\frac{d_i}{r}\right) + \frac{1}{2}\left(3\cos^2(\theta_i) - 1\right)\left(\frac{d_i}{r}\right)^2 + \ldots\right] = \frac{1}{4\pi\varepsilon_0}q_i\left[\frac{1}{r} + \frac{k_{i2}}{r^2} + \frac{k_{i3}}{r^3} + \ldots\right]$$

Dipolo elettrico: sviluppo in multipoli



$$V_i = \frac{1}{4\pi\varepsilon_0}\frac{q_i}{r_i} =$$

$$= \frac{1}{4\pi\varepsilon_0} q_i \left(\frac{1}{r} + \frac{k_{i2}}{r^2} + \frac{k_{i3}}{r^3} + \ldots \right)$$

$$V = \sum_i V_i = \frac{1}{4\pi\varepsilon_0} \sum_i q_i \left(\frac{1}{r} + \frac{k_{i2}}{r^2} + \frac{k_{i3}}{r^3} + \ldots \right) =$$

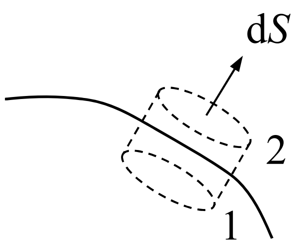
$$= \frac{1}{4\pi\varepsilon_0} \left[\frac{1}{r} \sum_i q_i + \frac{1}{r^2} \sum_i q_i k_{i2} + \frac{1}{r^3} \sum_i q_i k_{i3} + \ldots \right] =$$

$$= \frac{1}{4\pi\varepsilon_0} \left(\frac{k_1}{r} + \frac{k_2}{r^2} + \frac{k_3}{r^3} + \ldots \right)$$

termine di...

- k_1 ...monopolo
- k_2 ...dipolo
- k_3 ...quadrupolo

Formulazione differenziale: condizioni al contorno



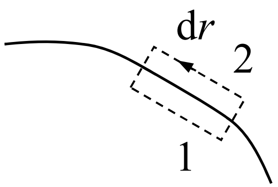
$$\begin{cases} d\Phi_2\left(\vec{E}\right) = \vec{E}_2 \cdot d\vec{S}_2 = E_{n2} dS \\ d\Phi_1\left(\vec{E}\right) = \vec{E}_1 \cdot d\vec{S}_1 = -\vec{E}_1 \cdot d\vec{S}_2 = -E_{n1} dS \\ d\Phi_{lat}\left(\vec{E}\right) \approx 0 \end{cases}$$

$$d\Phi\left(\vec{E}\right) = d\Phi_1\left(\vec{E}\right) + d\Phi_2\left(\vec{E}\right) + d\Phi_{lat}\left(\vec{E}\right) = \left(E_{n2} - E_{n1}\right) dS = \Delta E_n dS$$

$$d\Phi\left(\vec{E}\right) = \frac{dq}{\varepsilon_0} = \frac{\sigma \, dS}{\varepsilon_0}$$

$$\Delta E_n = \frac{\sigma}{\varepsilon_0}$$

Formulazione differenziale: condizioni al contorno



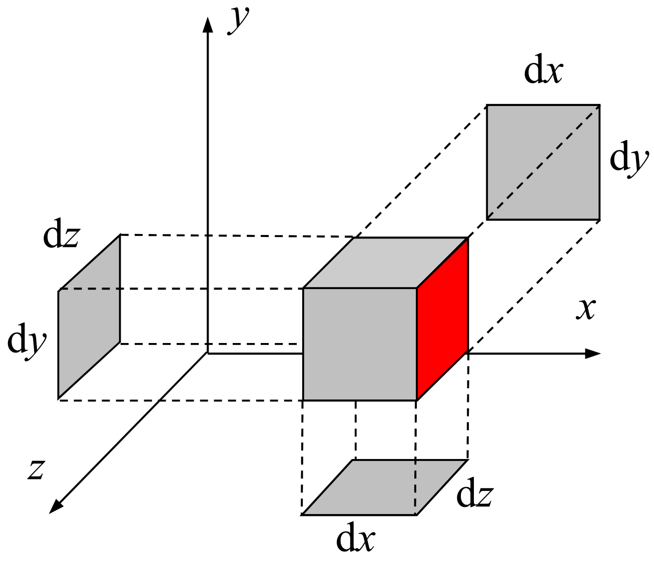
$$\begin{cases} d\Lambda_2\left(\vec{E}\right) = \vec{E}_2 \cdot d\vec{r}_2 = E_{t2} dr \\ d\Lambda_1\left(\vec{E}\right) = \vec{E}_1 \cdot d\vec{r}_1 = -\vec{E}_1 \cdot d\vec{r}_2 = -E_{t1} dr \\ d\Lambda_n\left(\vec{E}\right) \approx 0 \end{cases}$$

$$d\Lambda\left(\vec{E}\right) = d\Lambda_1\left(\vec{E}\right) + d\Lambda_2\left(\vec{E}\right) + d\Lambda_n\left(\vec{E}\right) = \left(E_{t2} - E_{t1}\right) dr = \Delta E_t dr$$

$$d\Lambda\left(\vec{E}\right) = 0$$

$$\Delta E_t = 0$$

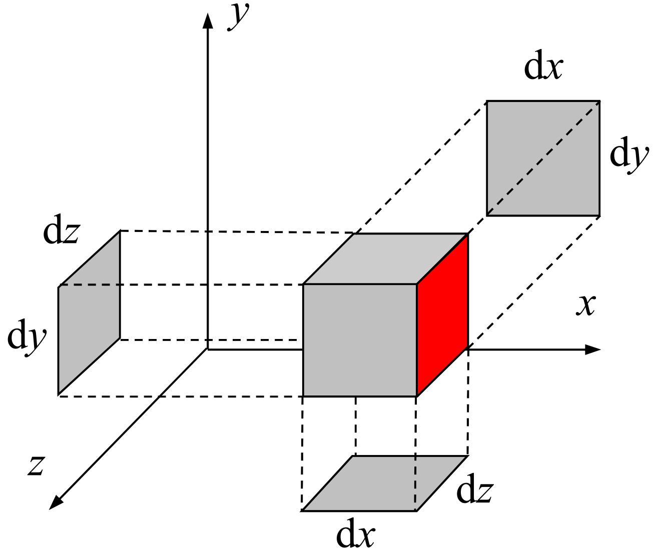
Formulazione differenziale: leggi di Maxwell



$$\begin{cases} d\Phi''_x\left(\vec{E}\right) = \vec{E}'' \cdot d\vec{S} = E''_x dS = E''_x dydz \\ d\Phi'_x\left(\vec{E}\right) = \vec{E}' \cdot d\vec{S} = -E'_x dS = -E'_x dydz \end{cases}$$

$$\begin{cases} d\Phi_x\left(\vec{E}\right) = d\Phi''_x\left(\vec{E}\right) + d\Phi'_x\left(\vec{E}\right) = E''_x dydz - E'_x dydz = dE_x dydz = \left(\frac{\partial E_x}{\partial x} dx\right) dydz = \frac{\partial E_x}{\partial x} dV \\ d\Phi_y\left(\vec{E}\right) = \frac{\partial E_y}{\partial y} dV \\ d\Phi_z\left(\vec{E}\right) = \frac{\partial E_z}{\partial z} dV \end{cases}$$

Formulazione differenziale: leggi di Maxwell



$$\Phi\left(\vec{E}\right) = \int \mathrm{d}\Phi\left(\vec{E}\right) = \oint \vec{E}\cdot \mathrm{d}\vec{S} = \int \mathrm{div}\left(\vec{E}\right)\mathrm{d}V$$

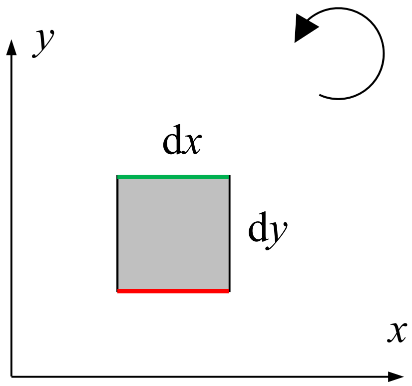
teorema della divergenza

$$\mathrm{d}\Phi\left(\vec{E}\right) = \mathrm{d}\Phi_x\left(\vec{E}\right) + \mathrm{d}\Phi_y\left(\vec{E}\right) + \mathrm{d}\Phi_z\left(\vec{E}\right) = \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right)\mathrm{d}V = \mathrm{div}\left(\vec{E}\right)\mathrm{d}V$$

$$\mathrm{d}\Phi\left(\vec{E}\right) = \frac{\mathrm{d}q}{\varepsilon_0} = \frac{\rho\,\mathrm{d}V}{\varepsilon_0}$$

$$\mathrm{div}\left(\vec{E}\right) = \frac{\rho}{\varepsilon_0}$$

Formulazione differenziale: leggi di Maxwell

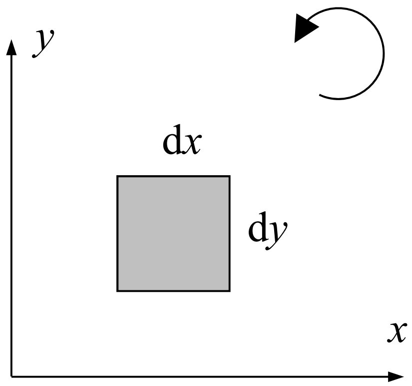


$$\begin{cases} d\Lambda_x'\left(\vec{E}\right) = \vec{E}' \cdot d\vec{r} = E_x' dx \\ d\Lambda_x''\left(\vec{E}\right) = \vec{E}'' \cdot d\vec{r} = -E_x'' dx \end{cases}$$

$$\begin{cases} d\Lambda_x\left(\vec{E}\right) = d\Lambda_x'\left(\vec{E}\right) + d\Lambda_x''\left(\vec{E}\right) = E_x' dx - E_x'' dx = -dE_x dx = -\left(\dfrac{\partial E_x}{\partial y} dy\right) dx \\ d\Lambda_y\left(\vec{E}\right) = dE_y dy = \dfrac{\partial E_y}{\partial x} dx dy \end{cases}$$

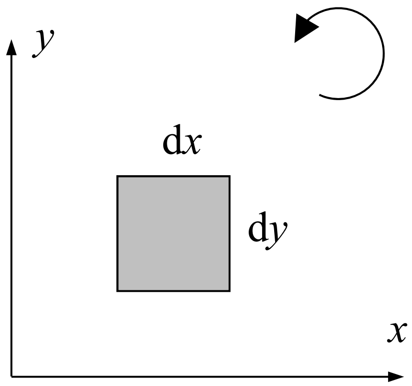
$$d\Lambda\left(\vec{E}\right) = d\Lambda_y\left(\vec{E}\right) + d\Lambda_x\left(\vec{E}\right) = \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) dx dy$$

Formulazione differenziale: leggi di Maxwell



$$\begin{cases} \text{piano } xy: \quad \mathrm{d}\Lambda\left(\vec{E}\right) = \left(\dfrac{\partial E_y}{\partial x} - \dfrac{\partial E_x}{\partial y}\right)\mathrm{d}x\mathrm{d}y = \left(\dfrac{\partial E_y}{\partial x} - \dfrac{\partial E_x}{\partial y}\right)\mathrm{d}S_z \\ \text{piano } yz: \quad \mathrm{d}\Lambda\left(\vec{E}\right) = \left(\dfrac{\partial E_z}{\partial y} - \dfrac{\partial E_y}{\partial z}\right)\mathrm{d}y\mathrm{d}z = \left(\dfrac{\partial E_z}{\partial y} - \dfrac{\partial E_y}{\partial z}\right)\mathrm{d}S_x \\ \text{piano } zx: \quad \mathrm{d}\Lambda\left(\vec{E}\right) = \left(\dfrac{\partial E_x}{\partial z} - \dfrac{\partial E_z}{\partial x}\right)\mathrm{d}z\mathrm{d}x = \left(\dfrac{\partial E_x}{\partial z} - \dfrac{\partial E_z}{\partial x}\right)\mathrm{d}S_y \end{cases}$$

Formulazione differenziale: leggi di Maxwell



$$\Lambda\left(\vec{E}\right) = \int \mathrm{d}\Lambda\left(\vec{E}\right) = \oint \vec{E}\cdot \mathrm{d}\vec{r} = \int \mathrm{rot}\left(\vec{E}\right)\cdot \mathrm{d}\vec{S}$$

teorema del rotore

$$\mathrm{d}\Lambda\left(\vec{E}\right) = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\mathrm{d}S_x + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\mathrm{d}S_y + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\mathrm{d}S_z = \mathrm{rot}\left(\vec{E}\right)\cdot \mathrm{d}\vec{S}$$

$$\mathrm{d}\Lambda\left(\vec{E}\right) = 0$$

$$\mathrm{rot}\left(\vec{E}\right) = 0$$

Formulazione differenziale: leggi di Maxwell

	Teorema di Gauss	Campo conservativo
relazioni integrali	$\Phi\left(\vec{E}\right) = \oint \vec{E} \cdot d\vec{S} = \frac{q_{int}}{\varepsilon_0}$	$\Lambda\left(\vec{E}\right) = \oint \vec{E} \cdot d\vec{r} = 0$
condizioni al contorno	$\Delta E_n = \frac{\sigma}{\varepsilon_0}$	$\Delta E_t = 0$
relazioni infinitesime	$\mathrm{div}\left(\vec{E}\right) = \frac{\rho}{\varepsilon_0}$	$\mathrm{rot}\left(\vec{E}\right) = 0$

Formulazione differenziale: leggi di Maxwell

$$\begin{cases} \mathrm{grad}(V) = \dfrac{\partial V}{\partial x}\vec{u}_x + \dfrac{\partial V}{\partial y}\vec{u}_y + \dfrac{\partial V}{\partial z}\vec{u}_z \\ \mathrm{div}(\vec{E}) = \dfrac{\partial E_x}{\partial x} + \dfrac{\partial E_y}{\partial y} + \dfrac{\partial E_z}{\partial z} \\ \mathrm{rot}(\vec{E}) = \begin{vmatrix} \vec{u}_x & \vec{u}_y & \vec{u}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} \end{cases} \Rightarrow \begin{cases} \mathrm{grad}(V) = \vec{\nabla} V \\ \mathrm{div}(\vec{E}) = \vec{\nabla} \cdot \vec{E} \\ \mathrm{rot}(\vec{E}) = \vec{\nabla} \times \vec{E} \end{cases}$$

$$\vec{\nabla} = \frac{\partial}{\partial x}\vec{u}_x + \frac{\partial}{\partial y}\vec{u}_y + \frac{\partial}{\partial z}\vec{u}_z$$

nabla

Formulazione differenziale: identità vettoriali

$$\begin{cases} \text{grad}(V) = \dfrac{\partial V}{\partial x}\vec{u}_x + \dfrac{\partial V}{\partial y}\vec{u}_y + \dfrac{\partial V}{\partial z}\vec{u}_z \\[2ex] \text{div}\left(\vec{E}\right) = \dfrac{\partial E_x}{\partial x} + \dfrac{\partial E_y}{\partial y} + \dfrac{\partial E_z}{\partial z} \\[2ex] \text{rot}\left(\vec{E}\right) = \begin{vmatrix} \vec{u}_x & \vec{u}_y & \vec{u}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} \end{cases}$$

derivate prime

$$\text{div}\left(\vec{A}\times\vec{B}\right) = \text{rot}\left(\vec{A}\right)\cdot\vec{B} - \vec{A}\cdot\text{rot}\left(\vec{B}\right)$$

$$\text{div}\left(f\vec{A}\right) = f\,\text{div}\left(\vec{A}\right) + \vec{A}\cdot\text{grad}(f)$$

$$\begin{cases} \text{rot}\left(\text{grad}(f)\right) \equiv 0 \\[2ex] \text{div}\left(\text{rot}\left(\vec{A}\right)\right) \equiv 0 \\[2ex] \nabla^2\left(\vec{A}\right) \equiv \text{grad}\left(\text{div}\left(\vec{A}\right)\right) - \text{rot}\left(\text{rot}\left(\vec{A}\right)\right) \end{cases}$$

derivate seconde

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

teorema di Schwar(t)z

Formulazione differenziale: leggi di Maxwell

$$\begin{cases} \mathrm{div}\left(\vec{E}\right) = \dfrac{\rho}{\varepsilon_0} \\ \mathrm{rot}\left(\vec{E}\right) = 0 \end{cases}$$

$$\mathrm{rot}\left(\mathrm{grad}\left(f\right)\right) \equiv 0 \quad \Rightarrow \quad \vec{E} = -\,\mathrm{grad}\left(V\right)$$

$$V = ?$$

$$V = \frac{1}{4\pi\varepsilon_0}\int\frac{\mathrm{d}q}{r} = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho}{r}\mathrm{d}V$$

vale la sovrapposizione degli effetti

Formulazione differenziale: leggi di Maxwell

$$\begin{cases} \mathrm{div}\left(\vec{E}\right) = \dfrac{\rho}{\varepsilon_0} \\ \mathrm{rot}\left(\vec{E}\right) = 0 \end{cases}$$

$$\mathrm{rot}\left(\mathrm{grad}\left(f\right)\right) \equiv 0 \quad \Rightarrow \quad \vec{E} = -\,\mathrm{grad}\left(V\right)$$

arbitraria

$$V = V_0 + k$$

$$\mathrm{grad}\left(V\right) = \mathrm{grad}\left(V_0 + k\right) =$$

$$= \mathrm{grad}\left(V_0\right) + \cancel{\mathrm{grad}\left(k\right)} = \mathrm{grad}\left(V_0\right)$$

$$-\,\mathrm{div}\left(\vec{E}\right) = -\,\mathrm{div}\left(-\,\mathrm{grad}\left(V\right)\right) = \nabla^2\left(V\right) = -\frac{\rho}{\varepsilon_0}$$

equazione di Poisson

$$\nabla^2 = \vec{\nabla} \cdot \vec{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

laplaciano

Formulazione differenziale: energia elettrica

$$\mathrm{div}\left(\vec{E}\right) = \frac{\rho}{\varepsilon_0} \qquad \mathrm{div}\left(f\vec{A}\right) = f\,\mathrm{div}\left(\vec{A}\right) + \vec{A}\cdot\mathrm{grad}\left(f\right) \qquad \vec{E} = -\,\mathrm{grad}\left(V\right)$$

$$\rho V = \varepsilon_0 V\,\mathrm{div}\left(\vec{E}\right) = \varepsilon_0\,\mathrm{div}\left(V\vec{E}\right) - \varepsilon_0\vec{E}\cdot\mathrm{grad}\left(V\right) = \varepsilon_0\,\mathrm{div}\left(V\vec{E}\right) + \varepsilon_0 E^2$$

$$E_{el} = \frac{1}{2}\int \rho V\,\mathrm{d}\tau = \frac{1}{2}\varepsilon_0\int \mathrm{div}\left(V\vec{E}\right)\mathrm{d}\tau + \frac{1}{2}\varepsilon_0\int E^2\mathrm{d}\tau \qquad E_{el} = \int \frac{1}{2}\varepsilon_0 E^2\mathrm{d}\tau$$

$$\int \mathrm{div}\left(V\vec{E}\right)\mathrm{d}\tau = \int V\vec{E}\,\mathrm{d}S$$

$$VE \propto \frac{1}{r}\frac{1}{r^2} \qquad S \propto r^2$$

$$\rho_{el} = \frac{1}{2}\varepsilon_0 E^2$$

densità di energia elettrica